Midterm — Analysis (WBMA012-05)

Friday 12 December 2025, 18.30h–20.30h

University of Groningen

Instructions

1. Students having a clash with another midterm, are allowed to take the exam from 16.00h to 18.00h in room 5161.0162 (Bernoulliborg). Note that it is **not** allowed to leave the room even when finishing earlier and it is **not** allowed to keep a copy of the exam or of the scrap paper.
2. The use of calculators, books, or notes is not allowed.
3. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may refer to all theorems and statements in the book, lectures or tutorials (unless differently and explicitly specified) but you should clearly indicate which of them you are using.
4. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (8 + 8 + 6 = 22 points)

Consider the following sequence:

$$x_{n+1} = \frac{1}{6 - 2x_n} \quad \text{with} \quad x_1 = 2.$$

(a) Show that ~~$x_n < x_{n+1}$~~ $x_n > x_{n+1}$ and $x_n > 0$ for all $n \in \mathbb{N}$.

(b) Prove that the sequence (x_n) converges and compute $\lim_{n\to\infty} x_n$.

(c) State the topological definition of convergence of a sequence, that is, the one in terms of ϵ-neighborhoods and explain its meaning in your own words.

Problem 2 (10 + 6 + 4 = 20 points)

Assume that $a_n \neq 0$ for all $n \in \mathbb{N}$ and $L = \lim \left|\frac{a_{n+1}}{a_n}\right|$ exists. Prove the following statements:

(a) For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$(L - \epsilon)^k |a_N| < |a_{N+k}| < (L + \epsilon)^k |a_N| \quad \text{for all } k \in \mathbb{N}.$$

Here the left inequality is only true if $\epsilon < L$. *This was not announced during the exam, so we will be lenient when grading.*

(b) $L < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolutely.

(c) $L > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

Please turn over for problems 3 and 4!

Problem 3 ($8 + 10 + 4 = 22$ points)

Let $K \subset \mathbb{R}$ be compact and nonempty.

(a) Show that $\sup K$ and $\inf K$ both exist.

(b) Show that $\sup K$ and $\inf K$ are elements of K.

(c) Is the set $\{1/n \mid n \in \mathbb{N}\}$ compact? Justify your answer.

Problem 4 ($8 + 6 + 12 = 26$ points)

Let $g : A \subset \mathbb{R} \to \mathbb{R}$. For any $B \subseteq \mathbb{R}$ define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in A : g(x) \in B\}.$$

(a) Write down the ϵ-δ definition of continuity of g at a point $c \in A$. How is this definition more general than claiming that $\lim_{x \to c} f(x) = f(c)$?

(b) Let $B, C \subseteq A$. Show that $g(C) \subseteq B$ if and only if $C \subseteq g^{-1}(B)$.

(c) *Let A be open.* Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

Hint: for the last point start by writing the definition of open sets and remember that you can use (b).

Note: The last point is the definition of continuity you will see in next year in Topology. At that point you will be able to allow more general domains A and will have a way to define what it means for a set to be open in A.

End of test (90 points)

Note that all the problems could be solved in multiple ways, and not all of those solutions are included here.

Solution of problem 1 (8 + 8 + 6 = 22 points)

(a) We have $x_2 = 1/2$, so $x_2 < x_1$. Now assume $x_{n+1} < x_n$ for some $n \in \mathbb{N}$. Then, $6 - 2x_{n+1} > 6 - 2x_n$ and thus,

$$x_{n+2} = \frac{1}{6 - 2x_{n+1}} < \frac{1}{6 - 2x_n} = x_{n+1}.$$

By induction, $x_{n+1} < x_n$ for all $n \in \mathbb{N}$.

Clearly $x_1 = 2 > 0$. Now assume $x_n > 0$ for some $n \in \mathbb{N}$. Then, $6 - 2x_n < 6$, and thus,

$$x_{n+1} = \frac{1}{6 - 2x_n} > \frac{1}{6} > 0.$$

(b) Since (x_n) is decreasing and bounded below by 0, it is convergent by the Monotone Convergence Theorem (MCT).

Let $L = \lim x_n$. Taking limits on both sides of the recurrence relation, we get

$$L = \frac{1}{6 - 2L}.$$

Multiplying both sides by $6 - 2L$ and rearranging, we obtain

$$2L^2 - 6L + 1 = 0.$$

Solving this quadratic equation we find

$$L = \frac{6 \pm \sqrt{36 - 8}}{4} = \frac{6 \pm \sqrt{28}}{4} = \frac{6 \pm 2\sqrt{7}}{4} = \frac{3 \pm \sqrt{7}}{2}.$$

Both roots are positive but we know the limit must be less than $x_2 = 1/2$ (since the sequence is decreasing), so we discard the larger root and conclude that

$$L = \frac{3 - \sqrt{7}}{2}.$$

(c) A sequence (x_n) converges to a limit L if it eventually gets arbitrarily close to it.

The word "eventually" is mathematically translated to the fact that there is a point in the sequence after which the property holds. In this case the property is being "arbitrarily close": this is made mathematically precise by stating that for any distance ϵ we can choose, all the terms in consideration are within that distance from L. You are also allowed to use a picture to illustrate this idea.

An ϵ-neighborhood of a point L is the open interval $V_\epsilon(L) = (L - \epsilon, L + \epsilon)$. The topological definition of convergence then states that a sequence (x_n) converges to L if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in V_\epsilon(L)$.

The latter is the same as requiring $|x_n - L| < \epsilon$ and thus the topological definition is literally the same as the ϵ-N definition of convergence:

for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$.

Solution of problem 2 (10 + 6 + 4 = 20 points)

(a) Let $\epsilon > 0$ be arbitrary, then there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies \left| \left| \frac{a_{n+1}}{a_n} \right| - L \right| < \epsilon.$$

Rewriting the inequality gives

$$n \geq N \implies (L - \epsilon)|a_n| < |a_{n+1}| < (L + \epsilon)|a_n|.$$

Setting $n = N$ proves the desired statement for $k = 1$. If the statement is true for some other $k \in \mathbb{N}$, then it follows that

$$N + k + 1 > N \implies |a_{N+k+1}| < (L + \epsilon)|a_{N+k}| < (L + \epsilon)(L + \epsilon)^k |a_N| = (L + \epsilon)^{k+1} |a_N|,$$

which proves the inequality for $k + 1$. The other inequality is proven similarly, *with the caveat that* $L - \epsilon > 0$. By induction, the statement holds for all $k \in \mathbb{N}$.

(b) If $L < 1$ we can take $0 < \epsilon < 1 - L$ so that $0 < L + \epsilon < 1$. By part (a) it follows that for n sufficiently large, the terms $|a_n|$ are bounded by the terms of a convergent geometric series. By the Comparison Test, it follows that $\sum_{n=N}^{\infty} |a_n|$ converges, which means that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(c) If $L > 1$ we can take $0 < \epsilon < L - 1$ so that $L - \epsilon > 1$. By part (a) it follows that the sequence (a_n) is unbounded. Therefore, $\sum_{n=1}^{\infty} a_n$ diverges.

Solution of problem 3 (8 + 10 + 4 = 22 points)

(a) Since K is compact it is bounded, so there exist real numbers m, M such that $m \leq x \leq M$ for all $x \in K$.

Thus M is an upper bound for K and m is a lower bound for K. By the Axiom of Completeness (and its equivalent version for the inf proved in the tutorials) $\sup K$ and $\inf K$ both exist.

Alternatively. As above, by the Axiom of Completeness $\sup K$ exists. Taking $-K := \{-x \mid x \in K\}$, since K is bounded below, $-K$ is bounded above, so $\sup(-K)$ exists. Proving the existence of $\inf K = -\sup(-K)$.

(b) Let $s = \sup K$, since s is the least upper bound for every $\epsilon > 0$ there exists an $x \in K$ with $s - \epsilon < x$. We want to show that $s \in K$. Note that this could be either an isolated point or a limit point of K.

If we can construct a sequence (x_n) in K that converges to s, then either $x_n = s$ for some n (possibly infinitely many) and thus $s \in K$, or there is a subsequence $x_{n_k} \neq s$ converging to s and thus s is a limit point of K. In this case, since K is compact, it is in particular closed and it contains all its limit points and thus $s \in K$.

Pick $\epsilon_n = 1/n$ and x_n such that $s - \epsilon_n < x_n$, we get that $(x_n) \to s$ since,

$$(s - \epsilon_n) \to s \quad \text{as} \quad n \to \infty.$$

Concluding this part of the proof.

The argument for the minimum is similar. Let $i = \inf K$, since i is the greatest lower bound for every $\epsilon > 0$ there exists an $x \in K$ with $x < i + \epsilon$. Pick $\epsilon_n = 1/n$ and x_n such that $x_n < i + \epsilon_n$, we get that $(x_n) \to i$ since,

$$(i + \epsilon_n) \to i \quad \text{as} \quad n \to \infty.$$

Then $i \in K$ by the same argument as above.

(c) No, since the set is not closed: 0 is its unique limit point (shown in different ways in class and tutorials) and is not included in the set.

Solution of problem 4 (8 + 6 + 12 = 26 points)

(a) For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ with $|x - c| < \delta$, it holds $|g(x) - g(c)| < \epsilon$.

This definition is more general than $\lim_{x \to c} g(x) = g(c)$ because it allows c to be an isolated point of A (where the limit may not make sense), and it only requires x to be in A.

(b) ($\implies$) Suppose $g(C) \subseteq B$. If $x \in C$, then $g(x) \in g(C) \subseteq B$, so $x \in g^{-1}(B)$. Thus $C \subseteq g^{-1}(B)$.

($\impliedby$) Conversely, suppose $C \subseteq g^{-1}(B)$. If $y \in g(C)$, then $y = g(x)$ for some $x \in C$, so $x \in g^{-1}(B)$ and $y = g(x) \in B$. Thus $g(C) \subseteq B$.

(c) ($\implies$) Suppose g is continuous, and let $O \subseteq \mathbb{R}$ be open.

For each $x \in g^{-1}(O)$, $g(x) \in O$, so there exists $\epsilon > 0$ such that $V_\epsilon(g(x)) \subseteq O$.

By continuity, there exists $\delta > 0$ such that for all $y \in A$ with $|x - y| < \delta$, it holds that $|g(x) - g(y)| < \epsilon$. That is, for all $y \in V_\delta(x) \cap A$, $g(y) \in V_\epsilon(g(x)) \subseteq O$. Or, more compactly, $g(V_\delta(x) \cap A) \subseteq V_\epsilon(g(x)) \subseteq O$.

This means that $V_\delta(x) \cap A \subseteq g^{-1}(O)$. Thus $g^{-1}(O)$ is open in A.

($\impliedby$) Suppose $g^{-1}(O)$ is open in A for every open $O \subseteq \mathbb{R}$.

Fix $c \in A$ and $\epsilon > 0$. Let $O = V_\epsilon(g(c))$, which is open.

Then $g^{-1}(O)$ is open in A and contains c, so there exists $\delta > 0$ such that $V_\delta(c) \cap A \subseteq g^{-1}(O)$.

Thus for all $x \in A$ with $|x - c| < \delta$, $g(x) \in O$, so $|g(x) - g(c)| < \epsilon$. Hence g is continuous at c.